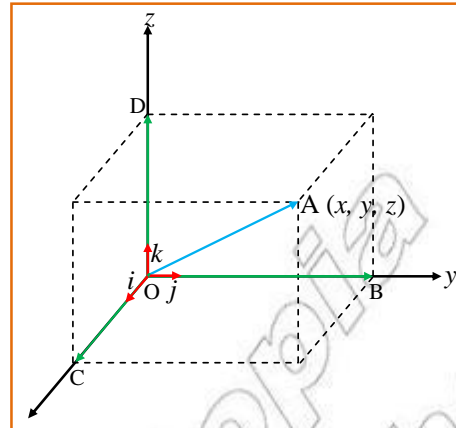# Unit 6



# THREE DIMENSIONAL GEOMETRY AND VECTORS IN SPACE

## Unit Outcomes:

*After completing this unit, you should be able to:*

➢ *know methods and procedures for setting up coordinate systems in space.*

➢ *know basic facts about coordinates and their use in determining geometric concepts in space.*

➢ *apply facts and principles about coordinates in space to solve related problems.*

➢ *know specific facts about vectors in space.*

## Main Contents

# INTRODUCTION

IN THIS UNIT, YOU WILL BE INTRODUCED TO THE COORDINATE SYSTEM IN SPACE WHICH IS AN ... OF THE COORDINATE SYSTEM ON THE PLANE THAT YOU ARE ALREADY FAMILIAR WITH. TH... BEGINS WITH A SHORT REVISION OF THE COORDINATE PLANE AND THEN INTRODUCES T... DIMENSIONAL COORDINATE SYSTEM. YOU WILL LEARN HOW THE THREE DIMENSIONAL COORDINA... USED TO FIND DISTANCE BETWEEN TWO POINTS, THE MIDPOINT OF A LINE SEGMENT IN SPACE ... HOW THEY ARE USED TO DERIVE THE EQUATION OF A SPHERE. FINALLY, YOU WILL SEE H... DIMENSIONAL COORDINATES CAN BE APPLIED TO THE STUDY OF VECTORS IN SPACE.

EACH TOPIC IN THIS UNIT IS PRECEDED BY A FEW ACTIVITIES AND YOU ARE EXPECTED TO ATTE... ACTIVITY. ATTEMPTING ALL THE EXERCISES AT THE END OF EACH SECTION WILL ALSO HELP Y... WITH CONFIDENCE.

## 📂 OPENING PROBLEM

TWO AIRPLANES TOOK OFF FROM THE SAME AIRPORT AT THE SAME TIME. ONE WAS HEADI... WITH A GROUND SPEED OF 800KM AND THE SECOND HEADING EAST WITH A GROUND SPEED OF ... 700KM/HR. IF THE FLIGHT LEVEL OF THE ONE HEADING NORTH IS 10KM AND THAT OF HEADI... IS 12KM, WHAT IS THE DIRECT DISTANCE BETWEEN THE TWO AIRPLANES EXACTLY ONE HO... TAKEOFF?

## 6.1 COORDINATE AXES AND COORDINATE PLANES IN SPACE

RECALL THAT YOU SET UP A RECTANGULAR COORDINATE SYSTEM ON A PLANE BY USING TWO ... THAT ARE PERPENDICULAR TO EACH OTHER AT A POINT O. ONE AXIS IS MADE, CALLED THE ... HORIZONTAL AND THE SECOND LINE IS MADE VERTICAL. THEN USING THESE TWO AXES ... YOU ASSOCIATE EACH POINT P OF THE PLANE WITH A UNIQUE ORDERED PAIR OF REAL NUMBE... AS $(x, y)$.
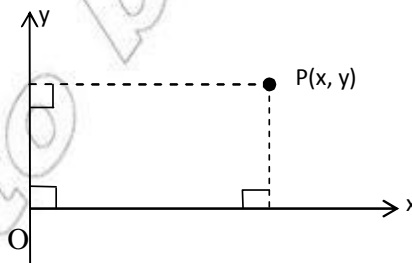


Figure 6.1

# ACTIVITY 6.1

1. PLOT EACH OF THE FOLLOWING POINTS ON THE COORDINATE PLANE.

  **A** $P(2, 3)$     **B** $Q(-3, 3)$     **C** $R(0, -4)$

  **D** $S(-2, -3)$     **E** $T\left(-\dfrac{3}{4}, \dfrac{1}{2}\right)$     **F** $U\left(\dfrac{3}{2}, -\dfrac{5}{2}\right)$

2. BY NAMING THE VERTICAL AXIS ON THE PLANE $z$ AND THE HORIZONTAL AXIS $x$, PLOT THE FOLLOWING POINTS.

  **A** $A(2, 4)$     **B** $B(-2, 3)$     **C** $C(-3, -4)$

THIS ASSOCIATION OF THE POINTS OF THE PLANE AND ORDERED PAIRS OF REAL NUMBERS IS ONE CORRESPONDENCE.

THE RECTANGULAR COORDINATE SYSTEM IS EXTENDED TO THREE DIMENSIONAL SPACES AS CONSIDER A FIXED POINT O IN SPACE AND THREE LINES THAT ARE MUTUALLY PERPENDICUL POINT O. THE POINT O IS CALLED THE ORIGIN; THE THREE LINES ARE NOW CALLED THE $x$-AXIS, $y$-AXIS AND THE $z$-AXIS. IT IS COMMON TO HAVE THE $x$ AND THE $y$ AXES ON A HORIZONTAL PLANE AND THE $z$-AXIS VERTICAL OR PERPENDICULAR TO THE PLANE CONTAINING THE AXES AT THE POINT O AS SHOWN IN FIGURE 6.2 BELOW. THE DIRECTIONS OF THE AXES ARE BASED ON THE RIGHT HAND RULE SHOWN IN FIGURE 6.2 BELOW.
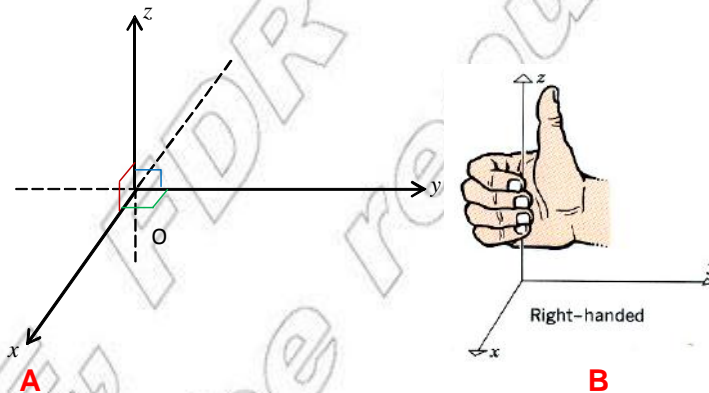


Figure 6.2

THE PLANE DETERMINED BY THE $x$ AND THE $y$ AXES IS CALLED $xy$-PLANE, THE PLANE DETERMINED BY THE $x$ AND THE $z$ AXES IS CALLED $xz$-PLANE AND THE PLANE DETERMINED BY THE $y$ AND THE $z$ AXES IS CALLED $yz$-PLANE. THESE THREE coordinate planes, WHICH INTERSECT AT THE ORIGIN, MAY BE VISUALIZED AS THE FLOOR OF A ROOM AND TWO ADJACENT WALLS OF THAT ROOM, FLOOR REPRESENTS $xy$-PLANE, THE TWO WALLS CORRESPONDING PLANES THAT INTERSECT ON $z$-AXIS AND THE CORNER OF THE ROOM CORRESPONDING TO THE ORIGIN.

COMMONLY, THE POSITIVE DIRECTION OF THE $x$-AXIS IS COMING OUT OF THE PAGE TOWARDS THE READER; THE POSITIVE $y$ DIRECTION IS TO THE RIGHT AND THE POSITIVE $z$ DIRECTION IS UPWARDS. (OPPOSITE DIRECTIONS TO THESE ARE NEGATIVE).

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NOTICE THAT THE coordinate planes PARTITION THE SPACE INTO EIGHT PARTS KNOWN AS octants. OCTANT 1 IS THE PART OF THE SPACE WHOSE BOUNDING EDGES ARE THE THREE POSITIVE NAMELY, THE POSITIVE $x$-AXIS, THE POSITIVE $y$-AXIS AND THE POSITIVE $z$-AXIS. THEN OCTANTS 2, 3 AND 4 ARE THOSE WHICH LIE ABOVE THE $xy$-PLANE IN THE COUNTER CLOCKWISE ORDER ABOUT THE $z$-AXIS. OCTANTS 5, 6, 7 AND 8 ARE THOSE WHICH LIE BELOW, WHERE OCTANT 5 IS JUST BELOW OCTANT 1 AND THE REST BEING IN THE COUNTER CLOCKWISE ORDER ABOUT THE $z$-AXIS.

## 6.2 COORDINATES OF A POINT IN SPACE

AS INDICATED AT THE BEGINNING OF THIS UNIT, A POINT P ON A PLANE IS ASSOCIATED WITH UNIQUE ORDERED PAIR OF REAL NUMBERS, USING TWO PERPENDICULAR LINES KNOWN AS THE $x$-AXIS AND THE $y$-AXIS. YOU ALSO REMEMBER $x$ REPRESENTS THE DIRECTED DISTANCE OF P FROM THE $y$-AXIS AND $y$ REPRESENTS THE DIRECTED DISTANCE OF P FROM $x$-AXIS. FOR EXAMPLE, IF THE COORDINATES OF P ARE (–3, 2), IT MEANS THAT P IS 3 UNITS left OF THE $y$-AXIS AND 2 UNITS above THE $x$-AXIS. SIMILARLY, A POINT Q(4, –5) IS FOUND 4 UNITS right OF THE $y$-AXIS AND 5 UNITS below THE $x$-AXIS.

### ACTIVITY 6.2

PLOT EACH OF THE FOLLOWING POINTS USING THE THREE AXES INTRODUCED ABOVE.

**A**  A(3,4,0)    **B**  B(0,3,4)    **C**  C(3,0,4)

NOW A POINT P IN SPACE IS LOCATED BY SPECIFYING ITS DIRECTED DISTANCES FROM THE COORDINATE PLANES. ITS DIRECTED DISTANCE FROM $yz$-PLANE MEASURED ALONG OR PARALLEL TO THE $x$-AXIS IS ITS $x$ COORDINATE. ITS DIRECTED DISTANCE FROM $xz$-PLANE MEASURED ALONG OR IN THE DIRECTION OF $y$-AXIS IS ITS $y$ COORDINATE AND ITS DIRECTED DISTANCE MEASURED ALONG OR IN THE DIRECTION OF $z$-AXIS FROM THE $xy$-PLANE IS ITS $z$ COORDINATE.

THE COORDINATES OF P ARE THEREFORE WRITTEN AS an ordered triple $(x, y, z)$ AS SHOWN IN FIGURE 6.3 BELOW.
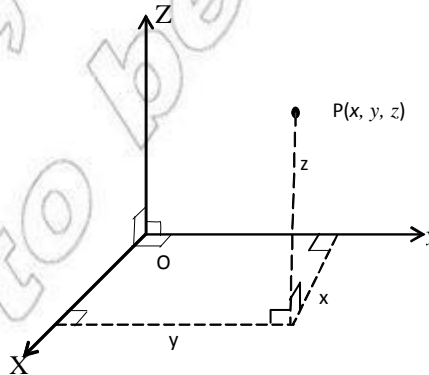


Figure 6.3

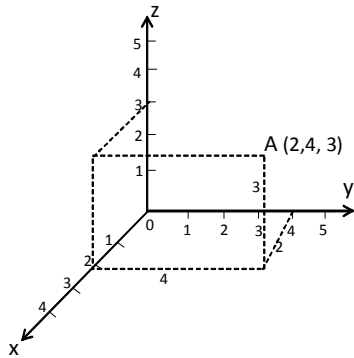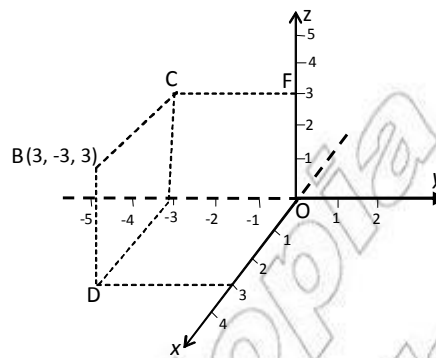**Example 1** LOCATE THE POINT A(2, 4, 3) IN SPACE USING THE REFERENCE AXES $x, y$ AND $z$.



Figure 6.4 (*Example 1*)



Figure 6.5 (*Example 2*)

**Example 2** LOCATE THE POINT B(3, -3, 3) IN SPACE USING THE REFERENCE AXES $x, y$ AND $z$.

THE PROCESS OF LOCATING THE POINT B MAY BE DESCRIBED AS FOLLOWS: START FROM ORIGIN O AND MOVE 3 UNITS IN THE DIRECTION OF THE POSITIVE $x$-AXIS THEN MOVE 3 UNITS IN THE DIRECTION OF THE NEGATIVE $y$-AXIS AND FINALLY MOVE 3 UNITS UP IN THE DIRECTION OF THE POSITIVE $z$-AXIS TO GET POINT B.

ON THE SAME COORDINATE SYSTEM OF EXAMPLE 2 ABOVE, NOTICE THAT THE COORDINATES OF POINT C ARE (0, -3, 3), THE COORDINATES OF POINT D ARE (3, -3, 0), THE COORDINATES OF POINT F ARE (0, 0, 3) AND THE COORDINATES OF POINT O (OR THE ORIGIN) ARE (0, 0, 0).

LOCATING A GIVEN POINT IN SPACE AS OBSERVED FROM THE DIFFERENT EXAMPLES ABOVE CAN BE CONSIDERED AS CORRESPONDING OR MATCHING A GIVEN ORDERED TRIPLE OF REAL NUMBER $(x, y, z)$ WITH SOME POINT P IN SPACE. (*SEE PROBLEMS 3, 4 and 5 of* EXERCISE 6.1.)

USING THIS FACT, IT IS POSSIBLE TO DESCRIBE SOME GEOMETRIC FIGURES IN SPACE BY MEANS OF EQUATIONS. FOR EXAMPLE, THE $x$-AXIS IS THE SET OF ALL POINTS IN SPACE $(x, y, z)$ WHOSE $y$ AND $z$ COORDINATES ARE ZERO. THUS WE EXPRESS IT AS FOLLOWS:

$x$-AXIS $= \{(x, y, z): x, y, z \in \mathbb{R} \text{ AND } y = z = 0\}$

## Exercise 6.1

**1** LOCATE EACH OF THE FOLLOWING POINTS IN SPACE USING REFERENCE AXES. YOU MAY USE THE SAME OR DIFFERENT COORDINATE SYSTEMS IN EACH CASE.

  **A**   P (3, 2, 3)     **B**   Q (–2, 4, 3)     **C**   R (3, – 3, 4)

  **D**   T(–2, –3,3)     **E**   M(0,0, –4)     **F**   N(2.5, $-\dfrac{1}{2}$ ,–3)

  **G**   Q(0,–3,0)

**2** GIVE THE EQUATIONS OF

  **A**   THE $y$-AXIS     **B**   THE $z$-AXIS     **C**   THE $xy$-PLANE

  **D**   THE $xz$-PLANE     **E**   THE $yz$-PLANE

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**3** GIVEN ANY POINT P IN SPACE, DRAW A COORDINATE SYSTEM AND

- ➤ DROP A PERPENDICULAR LINE TO THE PLANE.
- ➤ MARK THE INTERSECTION POINT OF THE PERPENDICULAR AND THE PLANE BY Q.
- ➤ MEASURE THE DISTANCE FROM P TO Q WITH A RULER AND MARK VALUE ON THE $z$- AXIS. CALL IT $z$.
- ➤ DROP PERPENDICULAR LINES TO $x$ AND $y$ AXES FROM POINT Q AND MARK THE INTERSECTION POINTS ON $x$ AND $y$ AXES, SAY
- ➤ THE TRIPLE $(x, y, z)$ YOU FOUND IN THE ABOVE STEPS UNIQUELY CORRESPONDS TO THE POINT. VERIFY! THUS $(x, y, z)$ ARE THE COORDINATE OF P IN SPACE.

**4** GIVEN ANY ORDERED TRIPLE $(a, b, c)$

- ➤ DRAW A COORDINATE SPACE AND LABEL EACH AXIS.
- ➤ MARK $a$ ON THE $x$ AXIS, $b$ ON THE $y$ AXIS AND $c$ ON THE $z$ AXIS.
- ➤ FROM $a$, DRAW A LINE PARALLEL TO $y$ AXIS AND, FROM $b$, DRAW A LINE PARALLEL TO THE $x$-AXIS; FIND THE INTERSECTION OF THE TWO LINES: MARK IT AS POINT R.
- ➤ FROM POINT R, DRAW A LINE PARALLEL TO $z$ AXIS, AND FROM $c$ DRAW A LINE PERPENDICULAR TO $z$ AXIS THAT INTERSECTS THE LINE FROM R. MARK THE INTERSECTION OF THESE TWO LINES BY POINT P.
- ➤ THE POINT P IN SPACE CORRESPONDS TO THE ORDERED TRIPLE. VERIFY THAT THERE IS NO OTHER POINT IN SPACE DESCRIBING THE ORDERED TRIPLE $(a, b, c)$. THUS $(a, b, c)$ ARE THE COORDINATES OF P.

**5** CAN YOU CONCLUDE FROM THE ABOVE TWO PROBLEMS, 3 AND 4, THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN THE SET OF POINTS IN SPACE AND THE SET OF ORDERED TRIPLES OF REAL NUMBERS? WHY? YOU MAY NEED TO USE THE BASIC FACTS OF SOLID GEOMETRY ABOUT PARALLEL AND PERPENDICULAR LINES AND PLANES IN SPACE.

## 6.3 DISTANCE BETWEEN TWO POINTS IN SPACE

### 📂 OPENING PROBLEM

ASSUME THAT YOUR CLASSROOM IS A RECTANGULAR BOX WHERE THE FLOOR IS 8 METRES AND METRES WIDE. IF THE DISTANCE FROM THE FLOOR TO THE CEILING (HEIGHT OF THE ROOM) METRES, FIND THE DIAGONAL DISTANCE BETWEEN A CORNER OF THE ROOM ON THE FLOOR OPPOSITE CORNER ON THE CEILING.

AFTER COMPLETING THIS SECTION, YOU WILL SEE THAT SOLVING THIS PROBLEM IS A MATTER OF FINDING DISTANCE BETWEEN TWO POINTS IN SPACE USING THEIR COORDINATES.

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# $A$CTIVITY 6.3

**1** ON THE COORDINATE PLANE, CONSIDER POINT $P(x_1, y_1)$ AND $Q(x_2, y_2)$ TO BE ANY TWO DISTINCT POINTS. THEN FIND THE DISTANCE BETWEEN P AND Q OR THE LENGTH OF THE LINE SEGMENT PQ BY USING THE PYTHAGORAS THEOREM.
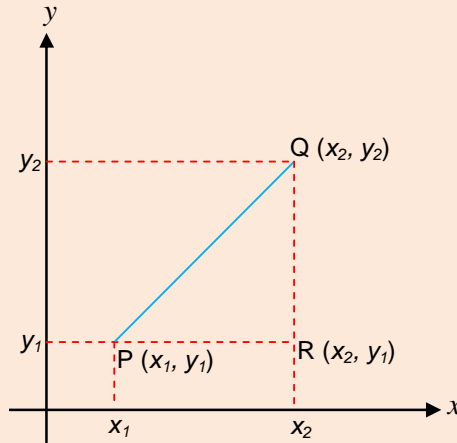


Figure 6.6

**2** FIND THE DISTANCE BETWEEN THE FOLLOWING PAIRS OF POINTS.

**A** A(3,4,0) AND B(1,5,0)    **B** C(0,3,4) AND D(0,1,2)

**C** E(4,0,5) AND F(1,0,1)

THE SAME PRINCIPLE WHICH YOU USE IN TWO DIMENSIONS CAN BE USED TO FIND THE DISTANCE BETWEEN TWO POINTS IN SPACE WHOSE COORDINATES ARE GIVEN.

FIRST, LET US CONSIDER THE DISTANCE OF $P(x, y, z)$ FROM THE ORIGIN O OF THE COORDINATE SYSTEM.

FROM THE POINT $P(x, y, z)$, LET US DROP PERPENDICULAR LINE SEGMENTS TO THE THREE PLANES AND LET US COMPLETE THE RECTANGULAR BOX, WHOSE EDGES ARE $x, y$ AND $z$ LONG AS SHOWN IN <span style="color:red">FIGURE 6.7</span> LET ITS VERTICES BE NAMED O, A,B,C, D, P, Q AND R.

THEN, TO FIND THE DISTANCE FROM O TO POINT P, CONSIDER THE RIGHT ANGLED TRIANGLE AND OBP.

HERE NOTICE THAT $\overline{PB}$ IS PERPENDICULAR TO $xy$-PLANE AT B, AND HENCE IT IS PERPENDICULAR TO $\overline{OB}$ AT B.

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Figure 6.7

NOW AS $\overline{OP}$ IS THE HYPOTENUSE OF THE RIGHT ANGLED TRIANGLE OBP, YOU KNOW B
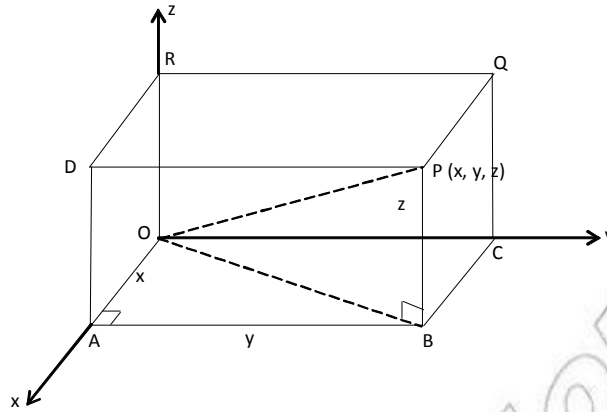PYTHAGORAS THEOREM THAT $(OP)^2 = (OB)^2 + (PB)^2$

ONCE AGAIN, $\overline{OB}$ IS THE HYPOTENUSE OF THE RIGHT ANGLED TRIANGLE OAB, YOU HAVE

$$(OB)^2 = (OA)^2 + (AB)^2.$$

THEN SUBSTITUTING $(OB)^2$ BY $(OA)^2 + (AB)^2$ IN $(OP)^2 = (OB)^2 + (PB)^2$, YOU OBTAIN

$$(OP)^2 = (OA)^2 + (AB)^2 + (PB)^2 = x^2 + y^2 + z^2$$

OR    $OP = \sqrt{x^2 + y^2 + z^2}$

---

✍**Note:**

OBSERVE THAT $\overline{OP}$ IS A DIAGONAL OF THE RECTANGULAR BOX AND $x$, $y$, $z$ IN ABSOLUTE VALUE,
ARE THE LENGTHS OF ITS THREE CONCURRENT EDGES. THEREFORE, THE DISTANCE FROM
NOW THE LENGTH OF THE DIAGONAL OF THE RECTANGULAR BOX WHICH IS THE SQUARE I
SUM OF THE SQUARES OF THE LENGTHS OF THE THREE EDGES OF THE BOX

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**Example 1**    FIND THE DISTANCE FROM THE ORIGIN TO THE POINT P(3, 4, 5).

**Solution**    THE DISTANCE FROM THE ORIGIN TO THE POINT P IS THE LENGTH OF THE LINE
SEGMENT $\overline{OP}$, WHICH IS

$$OP = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2} \text{ UNITS}$$

**Example 2**    FIND THE DISTANCE FROM THE ORIGIN TO THE POINT $Q(-2, 0, 3)$

**Solution** $OQ = \sqrt{(-2)^2 + 0^2 + 3^2} = \sqrt{13}$ UNITS

NOW, LET $P(x_1, y_1, z_1)$ AND $Q(x_2, y_2, z_2)$ BE ANY TWO POINTS IN SPACE. TO FIND THE DISTANCE
BETWEEN THESE TWO GIVEN POINTS, YOU MAY CONSIDER A RECTANGULAR BOX IN THE CO
SPACE SO THAT THE GIVEN POINTS P AND Q ARE ITS OPPOSITE VERTICES OR $\overline{PQ}$ IS ITS DIAGONAL AS
SHOWN IN FIGURE 6.8

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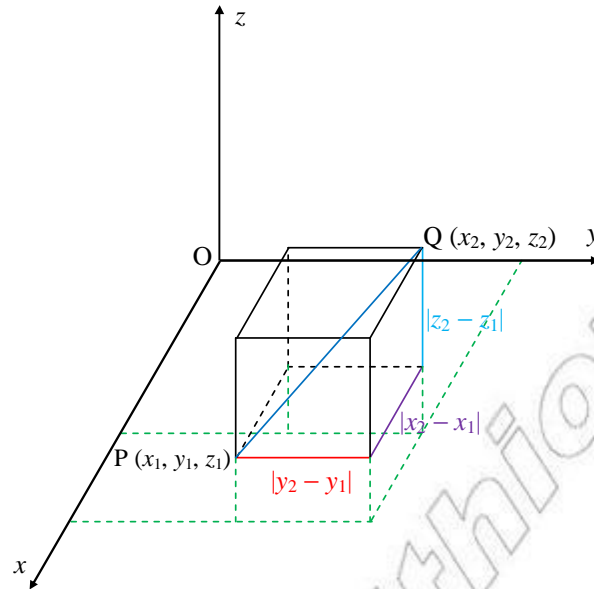
Figure 6.8

THEN WE SEE THAT THE LENGTHS OF THE THREE CONCURRENT EDGES OF THE BOX ARE $|x_2 - x_1|, |y_2 - y_1|$ AND $|z_2 - z_1|$.

THUS, THE DISTANCE FROM P TO Q OR THE LENGTH $\overline{PQ}$ OF THE DIAGONAL OF THE BOX IS GIVEN BY

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Example 3** FIND THE DISTANCE BETWEEN THE POINTS $P(1,-2,3)$ AND $Q(-4,0,5)$

**Solution** $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(-4-1)^2 + (0-(-2))^2 + (5-3)^2}$

$$= \sqrt{25 + 4 + 4} = \sqrt{33} \text{ UNITS}$$

## Exercise 6.2

**1** FIND THE DISTANCE BETWEEN THE GIVEN POINTS IN SPACE.

**A** A(0,1,0) AND B(2,0,3)  **B** C(2,1,3) AND D(4,6,10)

**C** E(–1,–3,6) AND F(4,0,–2)  **D** G(7,0,0) AND H(0,–4,2)

**E** $L\left(-1, -\dfrac{1}{2}, -\dfrac{1}{4}\right)$ AND M(–4,0,–1)  **F** N(7,11,12) AND P(–6,–2,0)

**G** $Q\left(\sqrt{2}, -\sqrt{2}, 1\right)$ AND R(0,0,–11)

**2** CAN YOU NOW SOLVE THE OPENING PROBLEM? PLEASE TRY IT.

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## 6.4 MIDPOINT OF A LINE SEGMENT IN SPACE

### ACTIVITY 6.4

ON THE COORDINATE PLANE, IF $P(x_1, y_1)$ AND $Q(x_2, y_2)$ ARE THE ENDPOINTS OF A LINE SEGMENT, YOU KNOW THAT ITS MIDPOINT M HAS COORDINATES $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
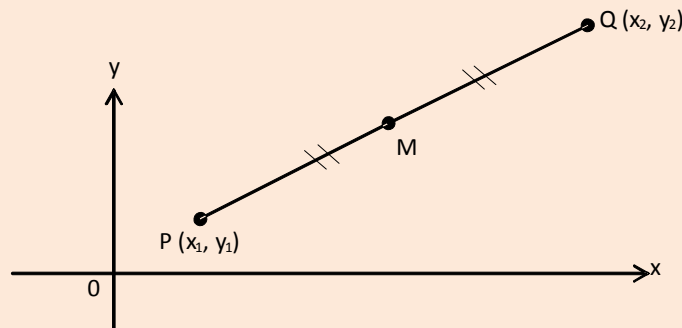


Figure 6.9

**1** FIND THE COORDINATES OF THE MIDPOINTS OF THE LINE SEGMENTS WITH GIVEN END POINTS ON A PLANE.

 **A**  A(2,4) AND B(0,2)     **B**  C(–1,3) AND D(3,–1)

 **C**  $E\left(\dfrac{1}{4}, -\dfrac{3}{4}\right)$ AND $F\left(\dfrac{3}{4}, \dfrac{3}{4}\right)$

**2** FIND THE COORDINATES OF THE MIDPOINTS OF THE LINE SEGMENTS WITH GIVEN END POINTS IN SPACE.

 **A**  A(2,4,0) AND B(0,2,0)     **B**  C (–1,3,0) AND D(3,–1,0)

 **C**  $E\left(\dfrac{1}{4}, -\dfrac{3}{4}, 0\right)$ AND $F\left(\dfrac{3}{4}, \dfrac{3}{4}, 0\right)$

THE COORDINATES OF THE MIDPOINT OF A LINE SEGMENT IN SPACE ARE ALSO OBTAINED IN THE WAY. THAT IS, THE COORDINATES OF THE MIDPOINT ARE OBTAINED BY TAKING THE AVERAGE RESPECTIVE COORDINATES OF THE ENDPOINTS OF THE GIVEN LINE SEGMENT. THUS, IF $P(x_1, y_1, z_1)$ AND $Q(x_2, y_2, z_2)$ ARE THE END POINTS OF A LINE SEGMENT IN SPACE, THE COORDINATES OF MIDPOINT M WILL BE $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$ See FIGURE 6.10
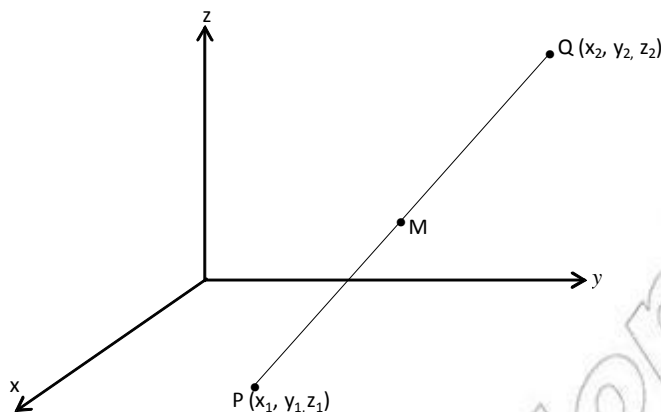
Figure 6.10

**Example 1** FIND THE MIDPOINT OF THE LINE SEGMENT WITH ENDPOINTS $A(0, 0, 0)$ AND B(4, 6, 2).

**Solution** THE MIDPOINT OF $\overline{AB}$ WILL BE AT THE POINT M WHOSE COORDINATES ARE

$$\left(\frac{0+4}{2}, \frac{0+6}{2}, \frac{0+2}{2}\right) = (2, 3, 1).$$

THAT IS, M(2, 3, 1) IS THE MIDPOINT OF $\overline{AB}$

**Example 2** FIND THE MIDPOINT OF THE LINE SEGMENT WHOSE ENDPOINT ARE $P(-1, 3, -3)$ AND Q(1, 5, 7).

**Solution** THE MIDPOINT OF $\overline{PQ}$ IS AT THE POINT M WHOSE COORDINATES ARE

$$\left(\frac{-1+1}{2}, \frac{3+5}{2}, \frac{-3+7}{2}\right) = (0, 4, 2).$$

SO, THE POINT M(0, 4, 2) IS THE MIDPOINT OF $\overline{PQ}$

## Exercise 6.3

**1** FIND THE MIDPOINT OF THE LINE SEGMENT WHOSE ENDPOIN ARE:

**A** A (1, 3, 5) AND B (3, 1, 1)    **B** P (0, –2, 2) AND Q (–4, 2, 4)

**C** $C\left(\frac{1}{2}, 3, 0\right)$ AND $D\left(\frac{3}{4}, -1, 1\right)$    **D** R (0, 9, 0) AND S (0, 0, 8)

**E** T (–2, –3, –5) AND U (–1, –1, –7)  **F** G (6, 0, 0) AND H (0, –4, –2)

**G** $M\left(\frac{1}{2}, \frac{1}{3}, -1\right)$ AND $K\left(-\frac{1}{2}, 0, \frac{1}{4}\right)$

**2** IF THE MIDPOINT OF A LINE SEGMENT IS AT M(2, 5, –6) AND ONE OF ITS ENDPOINTS IS AT R (–3, 2, 4), FIND THE COORDINATES OF THE OTHER ENDPOINT.

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## 6.5    EQUATION OF SPHERE

### ACTIVITY 6.5

WHEN THE CENTRE OF A CIRCLE AND ITS RADIUS, THE EQUATION OF THE CIRCLE IS GIVEN BY $(x - h)^2 + (y - k)^2 = r^2$

HERE NOTICE THAT C(h, k) IS THE CENTRE AND P(x, y) IS ANY POINT ON THE CIRCLE, AND r IS THE RADIUS OF THE CIRCLE OR THE DISTANCE FROM THE CENTRE C(h, k).

NOW USING SIMILAR NOTIONS:

1.  DEFINE A SPHERE WHOSE RADIUS IS r AND WHOSE CENTRE IS AT (a, b, c).

2.  **A**    FIND THE EQUATION OF THE SPHERE WHOSE CENTRE IS AT THE ORIGIN, AND HAS RADIUS $r = 2$.

    **B**    IF A POINT P(x, y, z) IS ON THE SURFACE OF THIS SPHERE, WHAT IS THE DISTANCE OF P FROM THE CENTRE OF THE SPHERE?

3.  IF THE CENTRE OF A SPHERE IS AT THE ORIGIN AND ITS RADIUS IS 5, WHAT IS THE DISTANCE OF A POINT P(3,4,0) ON THE SURFACE OF THE SPHERE FROM THE ORIGIN?

NOW, LET US CONSIDER A SPHERE WHOSE CENTRE IS AT THE ORIGIN OF A COORDINATE SYSTEM WHOSE RADIUS IS r. THEN, IF P(x, y, z) IS ANY POINT ON THE SURFACE OF THE SPHERE, THE LENGTH OF $\overline{OP}$ IS THE RADIUS OF THAT SPHERE. IN THE DISCUSSION ABOVE, YOU HAVE SEEN THAT LENGTH OF $\overline{OP}$ IS GIVEN BY $\sqrt{x^2 + y^2 + z^2}$. THEREFORE, EVERY POINT P(x, y, z) ON THE SPHERE SATISFIES THE EQUATION $r = \sqrt{x^2 + y^2 + z^2}$.

THAT MEANS, IF THE CENTRE OF A SPHERE IS AT THE ORIGIN OF THE COORDINATE SPACE AND r IS ITS RADIUS, THE EQUATION OF SUCH A SPHERE IS GIVEN BY $x^2 + y^2 + z^2 = r^2$
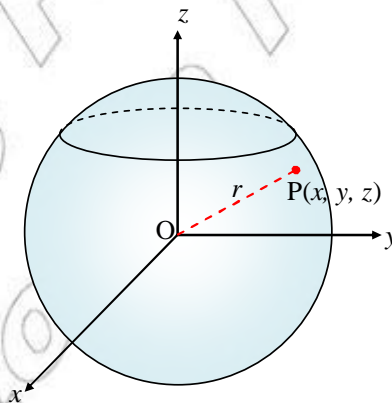


Figure 6.11

**Example 1**    WRITE THE EQUATION OF THE SPHERE WHOSE CENTRE IS AT THE ORIGIN AND WHOSE RADIUS IS 3 UNITS.

**Solution** IF P$(x, y, z)$ IS ANY POINT ON THE SPHERE, ITS DISTANCE FROM THE ORIGIN (THE CENTRE) IS GIVEN BY $r = \sqrt{x^2 + y^2 + z^2}$ . SUBSTITUTING $r$ BY 3, WE GET THE EQUATION OF THE SPHERE TO BE $\sqrt{x^2 + y^2 + z^2} = 3$ , WHICH IS EQUIVALENT TO $x^2 + y^2 + z^2 = 9$ .

THEREFORE, THE EQUATION OF THE SPHERE WILL BE $x^2 + y^2 + z^2 = 9$.

NOW LET US CONSIDER A SPHERE WHOSE CENTRE IS NOT AT THE ORIGIN BUT AT ANY OTHER C$(a, b, c)$ IN SPACE. IF P$(x, y, z)$ IS ANY POINT ON THE SURFACE OF THE SPHERE, THEN THE RADIUS OF THE SPHERE WILL BE THE LENGTH OF $\overline{CP}$.

THAT MEANS, IN THIS CASE $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

THEREFORE, THE EQUATION OF THE SPHERE IN THIS CASE IS

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$
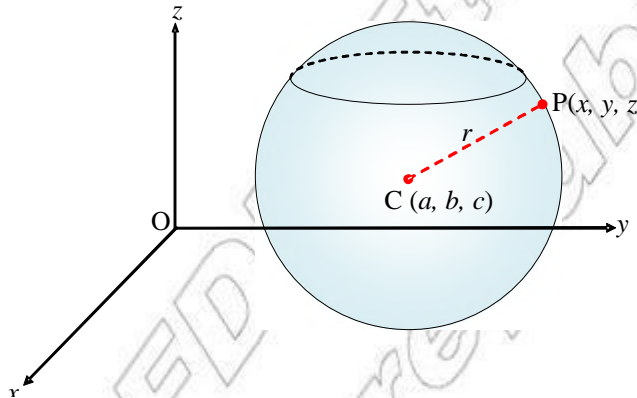


Figure 6.12

**Example 2** WRITE THE EQUATION OF THE SPHERE WITH CENTRE AT C$(1, 2, 3)$ AND RADIUS 4 UNITS.

**Solution** IF P$(x, y, z)$ IS ANY POINT ON THE SURFACE OF THE SPHERE, THEN THE DISTANCE FROM THE CENTRE C TO THE POINT P IS GIVEN TO BE THE RADIUS OF THE SPHERE.

THAT MEANS $r = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$

SUBSTITUTING $r$ BY 4 AND SQUARING BOTH SIDES, YOU GET THE EQUATION OF THE SPHERE TO BE

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 16 .$$

OBSERVE THAT WHEN THE CENTRE IS AT THE ORIGIN $(0, 0, 0)$ THE EQUATION

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \text{ REDUCES TO THE FORM } x^2 + y^2 + z^2 = r^2 .$$

(SUBSTITUTING $(a, b, c)$ BY $(0, 0, 0)$).

THAT MEANS THE EQUATION OF A SPHERE GIVEN BY $(x-a)^2+(y-b)^2+(z-c)^2=r^2$, WHERE r IS THE RADIUS AND (a, b, c) IS THE CENTRE CAN BE APPLIED TO A SPHERE WHOSE CENTRE IS AT ANY POINT (a, b, c) INCLUDING THE ORIGIN.

**Example 3** GIVEN THE EQUATION OF A SPHERE TO BE $x^2+y^2+z^2=9$, WHAT CAN YOU SAY ABOUT THE POINTS:

**A** P(1, 2, 2,)? **B** Q(0, 1, 2,)? **C** R(1, 3, 2)?

**Solution** CLEARLY THE CENTRE OF THE SPHERE IS AT THE ORIGIN AND ITS RADIUS IS 3.

**A** BECAUSE THE DISTANCE OF P FROM THE CENTRE IS 3, P IS ON THE SURFACE OF THE SPHERE.

**B** BECAUSE THE DISTANCE OF Q FROM THE CENTRE IS $\sqrt{5}$, WHICH IS LESS THAN 3, Q IS INSIDE THE SPHERE.

**C** BECAUSE THE DISTANCE OF R FROM THE CENTRE IS $\sqrt{14}$, R IS OUTSIDE THE SPHERE.

**In general, if O is the centre of a sphere and *r* is its radius, then for any point P taken in space, we have one of the following three possibilities.**

**I** OP = *r*, IN WHICH CASE P IS ON THE SURFACE OF THE SPHERE;

**II** OP < *r*, IN WHICH CASE P IS INSIDE THE SPHERE; AND

**III** OP > *r*, IN WHICH CASE P IS OUTSIDE THE SPHERE.

## Exercise 6.4

**1** WRITE THE EQUATION OF A SPHERE OF RADIUS 4 UNITS WHOSE CENTRE IS (3, 0, 5).

**2** GIVEN THE EQUATION OF A SPHERE TO BE $x^2+y^2+z^2-6x-4y-10z=-22$, FIND THE CENTRE AND RADIUS OF THE SPHERE.

**3** IF A(0, 0, 0) AND B(4, 6, 0) ARE END POINTS OF A DIAMETER OF A SPHERE, WRITE ITS EQUATION.

**4** HOW FAR IS THE POINT (1, 2, 3) FROM THE SPHERE WHOSE EQUATION IS
$(x-1)^2+(y+2)^2+z^2=1?$

**5** IF THE CENTRE OF A SPHERE IS AT THE ORIGIN AND ITS RADIUS IS 8, DETERMINE WHICH OF THE FOLLOWING POINTS LIE INSIDE OR OUTSIDE OR ON THE SPHERE.
A(2, 1, 2) B(–3, 2, 4) C(5, 8, 6) D(0, 8, 6) E(–8, –6, 0)

**6** DECIDE WHETHER OR NOT EACH OF THE FOLLOWING POINTS IS INSIDE, OUTSIDE, OR ON THE SPHERE WHOSE EQUATION IS
$x^2+y^2+z^2+2x-y+z=0.$

**A** O(0, 0, 0) **B** P(–1, 0, 1) **C** Q(0, $\frac{1}{2}$, 0)

**7** **A** STATE THE COORDINATES OF ANY POINT IN SPACE ON THE X-AXIS WHICH I
**B** FIND THE COORDINATES OF TWO POINTS ON THE X-AXIS WHICH ARE $\sqrt{2}$ UNITS FROM THE POINT P(–1, –1, 2).

## 6.6   VECTORS IN SPACE

RECALL THAT A VECTOR QUANTITY IS A QUANTITY THAT HAS BOTH MAGNITUDE AND DIRECTI
VELOCITY AND FORCE ARE EXAMPLES OF VECTOR QUANTITIES. ON THE OTHER HAND, A QUAN
MAGNITUDE ONLY BUT NO DIRECTION IS CALLED A SCALAR QUANTITY. FOR EXAMPLE, MASS AN
EXAMPLES OF SCALAR QUANTITIES.

### ACTIVITY 6.6

1    HOW DO YOU REPRESENT A VECTOR ON A PLANE?
2    HOW DO YOU REPRESENT THE MAGNITUDE OF A VECTOR?
3    HOW DO YOU SHOW THE DIRECTION OF A VECTOR?
4    HOW DO YOU EXPRESS THE VECTOR IN FIGURE 6.13 BELOW USING THE STANDARD UNIT
     VECTORS $\mathbf{i}$ AND $\mathbf{j}$?
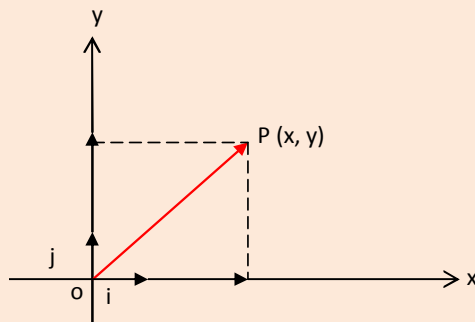


Figure 6.13

RECALL ALSO THAT THE VECTOR $\overrightarrow{OP}$ CAN BE NAMED USING A SINGLE LETTER, THAT IS, IT MAY BE
NAMED $\vec{a}$, OR SIMPLY AS $a$ SO THAT $\vec{a} = x\mathbf{i} + y\mathbf{j}$

OPERATIONS ON VECTORS CAN BE PERFORMED USING THEIR COMPONENTS OR THE COORD
THEIR TERMINAL POINTS WHEN THEIR INITIAL POINTS, ARE AT THE ORIGIN OF THE COORDINA

**Example 1**   IF $\vec{a} = 2\mathbf{i} + 4\mathbf{j}$ AND $\vec{b} = 5\mathbf{i} + 3\mathbf{j}$, THEN FIND

   **I**    $\vec{a} + \vec{b}$                    **II**    $\vec{a} - \vec{b}$

**Solution**

   **I**    $\vec{a} + \vec{b} = (2\mathbf{i} + 4\mathbf{j}) + (5\mathbf{i} + 3\mathbf{j}) = (2+5)\mathbf{i} + (4+3)\mathbf{j} = 7\mathbf{i} + 7\mathbf{j}$

   **II**   $\vec{a} - \vec{b} = (2\mathbf{i} + 4\mathbf{j}) - (5\mathbf{i} + 3\mathbf{j}) = (2-5)\mathbf{i} + (4-3)\mathbf{j} = -3\mathbf{i} + \mathbf{j}$

NOTICE THAT THE TERMINAL POINTS OF THE VECTORS $\vec{a}$ AND $\vec{b}$ ARE (2, 4) AND (5, 3) RESPECTIVELY,
WHILE THE TERMINAL POINT OF THE VECTOR $\vec{a} + \vec{b}$ IS (7, 7) WHICH CAN BE OBTAINED BY ADDING THE
CORRESPONDING COORDINATES OF THE TERMINAL POINTS OF $\vec{a}$ AND $\vec{b}$. THE TWO VECTORS
LOOK AS IN FIGURE 6.13 BELOW.

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Figure 6.14

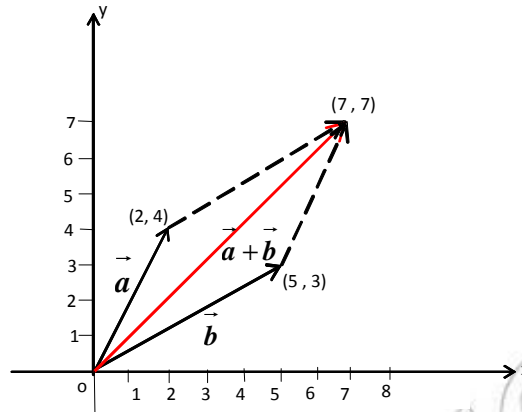IN YOUR PREVIOUS STUDIES, YOU HAVE ALSO LEARNED ABOUT THE SCALAR OR DOT PROD VECTORS. THAT IS, IF $\theta$ IS THE ANGLE BETWEEN THE TWO VECTORS $\vec{a}$ AND $\vec{b}$, THE DOT OR SCALAR PRODUCT OF $\vec{a}$ AND $\vec{b}$ DENOTED BY $\vec{a} \cdot \vec{b}$ IS DEFINED AS:

$(\vec{a}) \cdot (\vec{b}) = |\vec{a}||\vec{b}| \cos \theta$, WHERE $|\vec{a}|$ AND $|\vec{b}|$ ARE THE MAGNITUDES OF THE TWO VECTORS $\vec{a}$ AND $\vec{b}$ RESPECTIVELY.

**Example 2** COMPUTE THE SCALAR PRODUCT OF THE VECTORS $\vec{a} = 3\mathbf{i} + 3\mathbf{j}$ AND $\vec{b} = 4\mathbf{i} + 0\mathbf{j}$.

**Solution** BY PICTURING A DIAGRAM, THE ANGLE BETWEEN THE TWO VECTORS IS $45°$

THEN $|\vec{a}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ AND $|\vec{b}| = \sqrt{4^2 + 0^2} = 4$

THUS $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \theta = 3\sqrt{2}(4) \cos 45° = 12$.

OR

$\vec{a} \times \vec{b} = (3\mathbf{i} + 3\mathbf{j}) \cdot (4\mathbf{i} + 0\mathbf{j}) = (3 \times 4)\mathbf{i} \cdot \mathbf{i} + (3 \times 0)\mathbf{i} \cdot \mathbf{j} + (3 \times 4)\mathbf{j} \cdot \mathbf{i} + (3 \times 0)\mathbf{j} \cdot \mathbf{j}$

$= 12 + 0 + 0 + 0 = 12$.

## The notion of vectors in space

JUST AS YOU WORKED WITH VECTORS ON A PLANE BY USING THE COORDINATES OF THEIR POINTS, YOU CAN HANDLE VECTORS IN A THREE DIMENSIONAL SPACE WITH THE HELP COORDINATES OF THE TERMINAL POINTS.

NOW, LET THE INITIAL POINT OF A VECTOR IN SPACE BE THE ORIGIN O OF THE COORDINAT AND LET ITS TERMINAL POINT BE $A(x, y, z)$. THEN THE VECTOR $\overrightarrow{OA}$ CAN BE EXPRESSED AS THE SUM OF ITS THREE COMPONENTS IN THE DIRECTIONS OF THE $x, y$ AND $z$-AXIS, IN THE FORM:

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$\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$   WHERE $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ AND $\mathbf{k} = (0,0,1)$   ARE STANDARD UNIT VECTORS IN THE DIRECTIONS OF THE POSITIVE $x$, THE POSITIVE $y$ AND POSITIVE $z$ AXIS, RESPECTIVELY. LOOK AT FIGURE 6.15 BELOW.
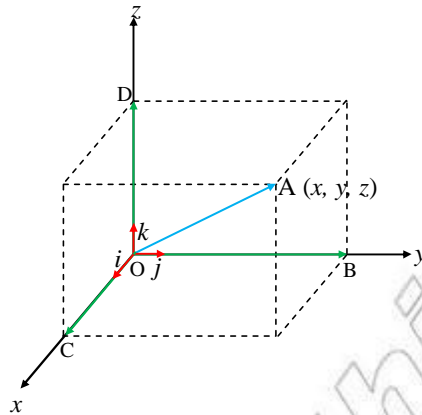


Figure 6.15

DO YOU OBSERVE THAT THE VECTOR $\overrightarrow{OA}$ IS THE SUM OF THE THREE PERPENDICULAR VECTORS $\overrightarrow{OC}, \overrightarrow{OB}$ AND $\overrightarrow{OD}$ ?

**Example 3**   IF THE INITIAL POINT OF A VECTOR IN SPACE IS AT THE ORIGIN AND ITS TERMINAL HEAD IS AT P(3, 5, 4), SHOW THE VECTOR USING A COORDINATE SYSTEM AND IDENTIFY ITS THREE PERPENDICULAR COMPONENTS IN THE DIRECTIONS OF THE THREE AXES.

**Solution**   THE THREE COMPONENTS ARE THE VECTORS WITH COMMON INITIAL POINT O(0, 0, 0) AND TERMINAL POINTS A(3, 0, 0) ON THE $x$-AXIS, B(0, 5, 0) ON THE $y$-AXIS AND C(0, 0, 4) ON THE $z$-AXIS AS SHOWN IN FIGURE 6.16



Figure 6.16

THAT MEANS, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ OR IN TERMS OF THE UNIT VECTORS,

$$(3, 5, 4) = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} = (3, 0, 0) + (0, 5, 0) + (0, 0, 4)$$

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## Addition and subtraction of vectors

JUST AS WITH VECTORS ON $xy$-PLANE, VECTORS IN SPACE CAN BE ADDED USING THE COORDINATES OF THEIR TERMINAL POINTS WHEN THEIR INITIAL POINTS ARE AT THE ORIGIN. IF $\vec{a}$ A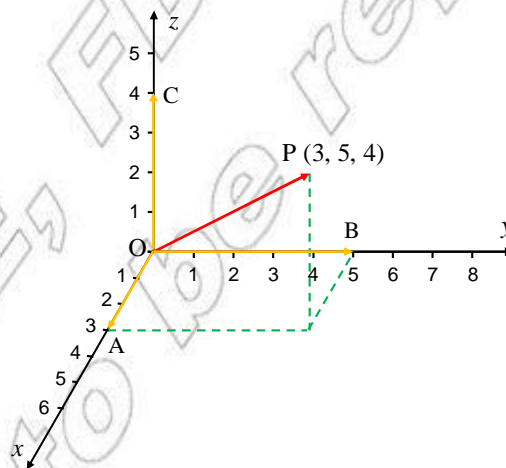ND $\vec{b}$ ARE VECTORS IN SPACE WITH THEIR INITIAL POINTS AT THE ORIGIN AND THEIR TERMINAL POINTS AT $(x_1, y_1, z_1)$ AND $(x_2, y_2, z_2)$, RESPECTIVELY, THEN $\vec{a} + \vec{b}$ IS THE VECTOR WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$.

HERE, IT IS ADVANTAGEOUS TO NOTE THAT A VECTOR $\vec{v}$ WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT POINT $P(x, y, z)$ IN SPACE IS SHORTLY EXPRESSED AS $\overrightarrow{OP} = \vec{v} = (x, y, z)$.

THUS $\vec{v} = (3, 2, -4)$ IS THE VECTOR IN SPACE WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT $(3, 2, -4)$.

**Example 4**  IF $\vec{a} = (1, 3, 2)$ AND $\vec{b} = (3, -1, 4)$, FIND THE SUM VECTOR $\vec{a} + \vec{b}$

**Solution**  AS EXPLAINED ABOVE, THE SUM OF THE TWO VECTORS IS OBTAINED BY ADDING THE CORRESPONDING COORDINATES OF THE TERMINAL POINTS OF THE TWO VECTORS.

THAT IS $\vec{a} + \vec{b} = (1, 3, 2) + (3, -1, 4) = (4, 2, 6)$ WHICH MEANS THAT $\vec{a} + \vec{b}$ IS THE VECTOR WHOSE INITIAL POINT IS THE ORIGIN AND WHOSE TERMINAL POINT IS AT (4,2,6).

SUBTRACTION OF A VECTOR FROM A VECTOR IS ALSO DONE IN A SIMILAR WAY. SO IF GIVEN TWO VECTORS $\vec{a} = (x_1, y_1, z_1)$ AND $\vec{b} = (x_2, y_2, z_2)$ THEN $\vec{a} - \vec{b}$ IS THE VECTOR $\vec{c} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$.

**Example 5**  IF $\vec{a} = (5, 2, 3)$ AND $\vec{b} = (3, 1, 4)$ THEN FIND $\vec{a} - \vec{b}$ AND $\vec{b} - \vec{a}$

**Solution**

**I**  $\vec{a} - \vec{b} = (5, 2, 3) - (3, 1, 4) = (5–3, 2–1, 3–4) = (2, 1, –1)$

THAT MEANS $\vec{a} - \vec{b}$ IS THE VECTOR WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT (2, 1, –1) IN SPACE.

**II**  $\vec{b} - \vec{a} = (3, 1, 4) - (5, 2, 3) = (3–5, 1–2, 4–3) = (-2, -1, 1)$.

DO YOU SEE THAT $\vec{a} - \vec{b} = -(\vec{b} - \vec{a})$?

## Multiplication of a vector by a scalar

IF $\vec{a} = (x, y, z)$ THEN OBSERVE THAT

$$2\vec{a} = \vec{a} + \vec{a} = (x, y, z) + (x, y, z) = (x + x, y + y, z + z) = (2x, 2y, 2z).$$

THUS, IT WILL BE REASONABLE TO ACCEPT THE FOLLOWING: IF $\vec{a} = (x, y, z)$ IS ANY VECTOR AND $k$ ANY SCALAR (A NUMBER), THEN

$k\vec{a} = (kx, ky, kz)$ WHICH IS A VECTOR WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT $(kx, ky, kz)$.

**Example 6** IF $\vec{a} = (4, 2, 3)$, THEN

**A** $\quad 3\vec{a} = (12, 6, 9)$ $\qquad$ **B** $\quad -\vec{a} = (-4, -2, -3)$

**C** $\quad -2\vec{a} = (-8, -4, -6)$ $\qquad$ **D** $\quad \dfrac{1}{2}\vec{a} = (2, 1, \dfrac{3}{2})$

## Properties of addition of vectors

SINCE VECTOR ADDITION IS DONE USING THE COORDINATES OR POINTS OF THE ADDEND VECTORS, WHICH ARE REAL NUMBERS, YOU CAN EASILY VERIFY THE FOLLOWING PROPERTI ADDITION.

**I** **Vector addition is commutative**

FOR ANY TWO VECTORS $\vec{a} = (x_1, y_1, z_1)$ AND $\vec{b} = (x_2, y_2, z_2)$ IN SPACE,

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ . TO SEE THIS, LET US LOOK AT THE FOLLOWING.

$\vec{a} + \vec{b} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2) = (x_2+x_1, y_2+y_1, z_2+z_1)$ *Why?*

$\qquad = (x_2, y_2, z_2) + (x_1, y_1, z_1) = \vec{b} + \vec{a}$

**II** **Vector addition is associative**

FOR ANY THREE VECTORS $\vec{a}$, $\vec{b}$ AND $\vec{c}$ IN SPACE, $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

**III** FOR TWO VECTORS $\vec{a}$ AND $\vec{b}$ AND ANY SCALAR $k$, YOU HAVE $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$ .

## Magnitude of a vector

AT THE BEGINNING OF THE DISCUSSION ABOUT VECTORS IN SPACE, WE MENTIONED THAT A VECTOR IS USUALLY REPRESENTED BY AN ARROW, WHERE THE ARROW HEAD INDICATES THE DIREC LENGTH OF THE ARROW REPRESENTS THE MAGNITUDE OF THE VECTOR. THUS, TO FIND THE OF A VECTOR, IT WILL BE SUFFICIENT TO FIND THE DISTANCE BETWEEN THE INITIAL POIN TERMINAL POINT OF THE VECTOR IN THE COORDINATE SPACE.

FOR EXAMPLE, IF THE INITIAL POINT OF A VECTOR IS AT THE ORIGIN OF THE COORDINATE S THE TERMINAL POINT IS AT P(3, 2, 4) THEN THE MAGNITUDE OF THE VECTOR $\overrightarrow{OP}$ IS THE DISTANCE FROM O TO P. THIS IS, AS YOU KNOW, $\sqrt{3^2 + 2^2 + 4^2} = \sqrt{29}$

THUS, IN GENERAL, IF THE INITIAL POINT OF A VECTOR IS AT THE ORIGIN AND ITS TERMINAL POINT IS AT A POINT $Q(x, y, z)$ OR IF $\vec{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , THEN MAGNITUDE OF THE VECTOR, DENOTED BY $|\vec{v}|$ IS GIVEN BY $\sqrt{x^2 + y^2 + z^2}$ . THAT IS,

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

IF THE INITIAL POINT OF $\vec{v}$ IS AT P$(x_1, y_1, z_1)$ AND THE TERMINAL POINT AT Q$(x_2, y_2, z_2)$, THEN

$$|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

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## Scalar or dot product of vectors in space

WHEN YOU WERE STUDYING VECTORS ON A PLANE, THE DOT PRODUCT (SCALAR PRODUCT) OF TWO VECTORS WAS DEFINED BY:

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos$ WHERE IS THE ANGLE BETWEEN THE TWO VECTORS. IN PARTICULAR, FOR T

UNIT VECTORS $\vec{i}$ AND $\vec{j}$, YOU KNOW THAT $|\vec{i}| = |\vec{j}| = 1$ AND FROM THE DEFINITION OF THE DOT

PRODUCT, YOU EASILY SEE THAT $\vec{i} \cdot \vec{j} = 0 = \vec{j} \cdot \vec{i}$. SO, IF $\vec{a} = (x_1, y_1)$, AND $\vec{b} = (x_2, y_2)$

THE DOT PRODUCT $\vec{a} \cdot \vec{b} = x_1.x_2 + y_1.y_2$ CAN BE VERIFIED VERY EASILY.

THE DOT (SCALAR) PRODUCT OF TWO VECTORS IN SPACE IS JUST AN EXTENSION OF THE DO
OF VECTORS ON A PLANE. THAT MEANS, IF $\vec{a}$ AND $\vec{b}$ ARE NOW TWO VECTORS IN SPACE, THE DOT
(SCALAR) PRODUCT OF $\vec{a}$ AND $\vec{b}$ DENOTED BY $\vec{a} \cdot \vec{b}$ IS DEFINED AS:

$\vec{a} \cdot \vec{b} = |\vec{a}|.|\vec{b}| \cos$ WHERE IS ONCE AGAIN THE ANGLE BETWEEN THE $\vec{a}$ AND $\vec{b}$ VECTORS.

OBSERVE THAT $\vec{a} \cdot \vec{b}$ IS A REAL NUMBER AND IN PARTICULAR IF

$\vec{a} = (x_1, y_1, z_1) = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ AND $\vec{b} = (x_2, y_2, z_2) = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, YOU SEE THAT,

THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION ENABLES YOU TO FIND:

$\vec{a} \cdot \vec{b} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \cdot (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) = x_1x_2 + y_1y_2 + z_1z_2$.

HERE IT IS IMPORTANT TO NOTE THAT FOR $\mathbf{i}, \mathbf{j}$ AND $\mathbf{k}$ UNIT VECTORS

$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ WHILE $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$ THE REASON BEING THAT THE MAGNITUDE OF A
UNIT VECTOR IS ONE, $\cos 0° = 1$ AND $\cos 90° = 0$.

**Example 7**  IF $\vec{a} = (2, 3, -1)$ AND $\vec{b} = (-1, 0, 2)$, THEN FIND THE SCALAR (DOT) PRODUCT OF
$\vec{a}$ AND $\vec{b}$.

**Solution**  $\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1z_2 = 2(-1) + 3(0) + (-1)(2) = -4$

**Example 8**  IF $\vec{a} = (2, 0, 2)$ AND $\vec{b} = (0, 3, 0)$ FIND THEIR DOT PRODUCT.

**Solution**  $\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1z_2 = 2(0) + 0(3) + 2(0) = 0$

OBSERVE THAT (2, 0, 2) AND (0, 3, 0) ARE PERPENDICULAR VECTORS I.E. THE ANGLE BETW
THEM IS $90°$.

## Exercise 6.5

**1**  CALCULATE THE MAGNITUDE OF EACH OF THE FOLLOWING VEC

  **A**  $(-1, 3, 0)$    **B**  $(3, 1, -1)$    **C**  $\left(\dfrac{1}{2}, \dfrac{3}{2}, \dfrac{4}{5}\right)$

**2**  FIND THE SCALAR (DOT) PRODUCT OF EACH OF THE PAIRS OF THE VECTORS.

  **A**  $(2, -3, 1)$ AND $(1, 0, 4)$    **B**  $(-5, 0, 1)$ AND $(1, -3, -2)$

  **C**  $(-2, 2, 0)$ AND $(0, 0, -1)$    **D**  $(0, 0, 3)$ AND $(0, 0, 3)$

## Angle between two vectors in space

FOR TWO VECTORS $\vec{a}$ AND $\vec{b}$ WITH INITIAL POINT AT THE ORIGIN, THEIR DOT PRODUCT IS DEFINED BY

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ WHERE $\theta$ IS THE ANGLE BETWEEN THE TWO VECTORS, ASSUMING THAT BOTH VECTORS HAVE THE SAME INITIAL POINT AT THE ORIGIN. THEN YOU CAN REWRITE $\cos\theta$ THE ABOVE EQUATION IN THE FORM:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

HENCE THE ANGLE $\theta$ BETWEEN THE TWO VECTORS CAN BE OBTAINED USING THIS LAST FORMULA, PROVIDED THE VECTORS ARE NON-ZERO.

**Example 9** FIND THE ANGLE BETWEEN THE VECTORS $\vec{a} = (2, 0, 0)$ AND $\vec{b} = (0, 0, 3)$.

**Solution** $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

BUT $\vec{a} \cdot \vec{b} = 2(0) + (0)(0) + 0(3) = 0$

$|\vec{a}| = \sqrt{2^2 + 0^2 + 0^2} = \sqrt{2}$ AND $|\vec{b}| = \sqrt{0^2 + 0^2 + 3^2} = 3$

THEREFORE $\cos\theta = \dfrac{0}{2(3)} = 0 \Rightarrow \theta = 90^\circ$

NOTICE THAT, THE VECTOR (2, 0, 0) IS ALONG THE $x$-AXIS WHILE THE VECTOR (0, 0, 3) IS ALONG THE $z$-AXIS AND THE TWO AXES ARE PERPENDICULAR TO EACH OTHER.

**Example 10** FIND THE ANGLE BETWEEN THE VECTORS $\vec{a} = (1, 0, 1)$ AND $\vec{b} = (1, 1, 0)$.

**Solution** $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

BUT $\vec{a} \cdot \vec{b} = 1(1) + 0(1) + 1(0) = 1$

$|\vec{a}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$ AND $|\vec{b}| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

THEREFORE $\cos\theta = \dfrac{1}{\sqrt{2}.\sqrt{2}} = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$

NOTICE THAT THE VECTOR $\vec{a}$ IS ON THE $xz$-PLANE AND $\vec{b}$ IS ON THE $xy$-PLANE, EACH FORMING A $45^\circ$ ANGLE WITH $x$-AXIS.

## Exercise 6.6

**1** IF THE VECTORS $\vec{a}, \vec{b}, \vec{v}$ AND $\vec{u}$ ARE AS GIVEN BELOW:

$\vec{a} = (1, 3, 2), \vec{b} = (0, -3, 4), \vec{v} = (-4, 3, -2), \vec{u} = (\frac{1}{2}, 0, -3)$, THEN FIND EACH OF THE

FOLLOWING VECTORS.

**A** $\vec{a} + \vec{b}$      **B** $\vec{b} + \vec{a}$      **C** $\vec{a} - \vec{b}$      **D** $\vec{b} - \vec{a}$

**E** $\vec{a} + \vec{b} + \vec{v}$      **F** $\vec{b} + \vec{v} - \vec{u}$      **G** $\vec{a} + \vec{b} + \vec{v} + \vec{u}$

**2** IF THE VECTORS $\vec{a}, \vec{b}, \vec{v}$ AND $\vec{u}$ ARE AS GIVEN QUESTION 1 ABOVE, THEN FIND

**A** $3\vec{a}$      **B** $-4\vec{b}$      **C** $2\vec{a} + 3\vec{b}$      **D** $3\vec{b} - \frac{1}{2}\vec{a} + 2\vec{v}$

**3** VERIFY THAT VECTOR ADDITION IS ASSOCIATIVE, THAT IS FOR ANY THREE VECTORS $\vec{a}, \vec{b}$ AND $\vec{c}$ IN SPACE, SHOW THAT $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

**4** FOR ANY TWO VECTORS $\vec{a}, \vec{b}$ IN SPACE AND ANY SCALAR $k$, SHOW THAT

$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

**5** WRITE EACH OF THE FOLLOWING VECTORS AS A SUM OF COMPONENTS USING THE STANDARD UNIT VECTORS $\vec{i}, \vec{j}$ AND $\vec{k}$.

**A** $(-4, 3, -2)$           **B** $(1, -3, \sqrt{2})$

**C** $(3, 5, -7)$           **D** $(0, 0, 3)$

**6** SHOW EACH OF THE FOLLOWING VECTORS IN SPACE USING AN ARROW THAT STARTS FROM THE ORIGIN.

**A** $\vec{a} = (3, 3, 3)$      **B** $\vec{b} = (-3, 3, 4)$      **C** $\vec{c} = (2, -3, -3)$

**7** CALCULATE THE MAGNITUDE OF EACH VECTOR IN QUESTION 6 ABOVE.

**8** FIND THE SCALAR (DOT) PRODUCT OF EACH OF THE FOLLOWING PAIRS OF VECTORS.

**A** $(1, 0, 1)$ AND $(2, 2, 0)$      **B** $(-2, 5, 1)$ AND $(1, -1, -2)$

**C** $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ AND $(-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2})$      **D** $(1, 0, 1)$ AND $(-1, 1, 0)$

**E** $(5, 0, 0)$ AND $(0, -5, 0)$      **F** $(2, 2, 2)$ AND $(-1, -1, -1)$

**9** FIND THE ANGLE BETWEEN EACH OF THE FOLLOWING PAIRS OF VECTORS

**A** $(2, 0, 1)$ AND $(0, -1, 0)$      **B** $(1, 1, 1)$ AND $(1, 0, 1)$

**C** $(-1, 1, 1)$ AND $(2, 2, 2)$

# Key Terms

# Summary

**1** THREE MUTUALLY PERPENDICULAR LINES IN SPACE DIVIDE THE SPACE INTO eight octants.

**2** IF $(x, y, z)$ ARE THE COORDINATES OF A POINT P IN SPACE, THEN

- ✓ $x$ IS THE DIRECTED DISTANCE OF THE POINT P FROM THE $yz$-PLANE.
- ✓ $y$ IS THE DIRECTED DISTANCE OF THE POINT P FROM THE $xz$-PLANE.
- ✓ $z$ IS THE DIRECTED DISTANCE OF THE POINT P FROM THE $xy$-PLANE.

**3** THERE IS A ONE TO ONE CORRESPONDENCE BETWEEN THE SET OF ALL POINTS OF THE SPACE AND THE SET OF ALL ORDERED TRIPLES OF REAL NUMBERS.

**4** THE DISTANCE BETWEEN TWO POINTS $P(x, y, z)$ AND $Q(a, b, c)$ IN SPACE IS GIVEN BY

$d = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ . THUS THE DISTANCE OF A POINT

$P(x, y, z)$ FROM THE ORIGIN IS $\sqrt{x^2 + y^2 + z^2}$ .

**5** THE midpoint OF A LINE SEGMENT WITH END POINTS $(x, y, z)$ AND $(a, b, c)$ IN SPACE

IS THE POINT $M \left( \dfrac{x+a}{2}, \dfrac{y+b}{2}, \dfrac{z+c}{2} \right)$

**6** THE equation of a sphere WITH CENTRE AT $C(x_1, y_1, z_1)$ AND RADIUS $r$ IS GIVEN BY

$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$

IN PARTICULAR, IF THE CENTRE IS AT THE ORIGIN AND $r$ IS ITS RADIUS, THE EQUATION BECOMES $x^2 + y^2 + z^2 = r^2$, WHERE $(x, y, z)$ ARE COORDINATES OF ANY POINT ON THE SURFACE OF THE SPHERE.

**7** IN SPACE, IF THE INITIAL POINT OF A VECTOR IS AT THE ORIGIN O OF THE COORDINATE AND ITS TERMINAL POINT IS AT A POINT $A(x, y, z)$, THEN IT CAN BE EXPRESSED AS THE SUM OF ITS THREE COMPONENTS IN THE DIRECTIONS OF THE THREE AXES AS:

$\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  WHERE $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ AND $\mathbf{k} = (0, 0, 1)$ ARE THE standard unit vectors IN THE DIRECTIONS OF THE POSITIVE $x$, POSITIVE $y$ AND POSITIVE $z$-AXES RESPECTIVELY.

**8** IN SPACE IF VECTORS $\vec{a}$ AND $\vec{b}$ HAVE THEIR INITIAL POINT AT THE ORIGIN AND THEIR TERMINAL POINTS AT A($x_1, y_1, z_1$) AND B($x_2, y_2, z_2$) RESPECTIVELY, THEN THEIR sum $\vec{a} + \vec{b}$ IS A VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN AND TERMINAL POINT IS AT C($x_1 + x_2, y_1 + y_2, z_1 + z_2$).

SIMILARLY, difference $\vec{a} - \vec{b}$ IS THE VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN AND TERMINAL POINT IS AT D($x_1 - x_2, y_1 - y_2, z_1 - z_2$).

**9** IF THE INITIAL POINT OF A VECTOR $\vec{a}$ IS THE ORIGIN AND THE TERMINAL POINT IS AT P($x, y, z$), THEN FOR ANY CONSTANT NUMBER $k$, THE PRODUCT $k\vec{a}$ IS A VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN AND TERMINAL POINT IS AT Q($kx, ky, kz$).

**10** VECTOR ADDITION IS COMMUTATIVE AND ALSO ASSOCIATIVE.

**11** MULTIPLICATION OF A VECTOR BY A SCALAR IS DISTRIBUTIVE OVER VECTOR ADDITION. FOR VECTORS $\vec{a}$ AND $\vec{b}$ AND A SCALAR $k$, $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$.

**12** THE MAGNITUDE OF A VECTOR $\vec{a}$ WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT P($x, y, z$) IS GIVEN BY:

$$|a| = \sqrt{x^2 + y^2 + z^2}$$

**13** THE dot (scalar) product OF TWO VECTORS $\vec{a}$ AND $\vec{b}$ IS GIVEN BY $\vec{b}.\vec{a} = |\vec{b}|.|\vec{a}|.\cos\theta$ WHERE $\theta$ IS THE ANGLE BETWEEN THE TWO VECTORS $\vec{a}$ AND $\vec{b}$.

**14** IF $\theta$ IS THE ANGLE BETWEEN TWO VECTORS $\vec{a}$ AND $\vec{b}$, THEN $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|./\vec{b}|}$.

## ? Review Exercises on Unit 6

**1** LOCATE THE POSITION OF EACH OF THE FOLLOWING POINTS AND FIND THE DISTANCE OF EACH POINT FROM THE ORIGIN.

**A** O(0,0, –3)   **B** P(0, –1, 2)   **C** Q(3,–1,4)

**D** R(–1,–2,–3)   **E** S(–3, –2, 3)   **F** T(–3, –3, –4)

**G** U(4, 3, –2)   **H** V $(0, -\dfrac{3}{2}, \dfrac{5}{2})$

**2** FIND THE DISTANCE OF THE POINT P (4, –3, 5) FROM THE

**A** ORIGIN   **B** $xy$-PLANE   **C** $xz$-PLANE   **D** $yz$-PLANE
**E** $x$-AXIS   **F** $y$-AXIS   **G** $z$-AXIS

**3** FOR EACH OF THE FOLLOWING PAIRS OF POINTS, FIND THE DISTANCE FROM A TO B AND FIND THE MIDPOINT OF $\overline{AB}$

**A** $A$ (–1,2,3) AND $B$ (0,–1,1)   **B** $A$ (3,–1,1) AND $B$ (–1, 0,1)

**C** $A$ (2,0,–3) AND $B$ (2,–1,3)   **D** $A$ (0,0,–4) AND $B$ (4,0,0)

**E**  $A$ (–2, –1, –3) AND $B$ (2, 2, 1)   **F**  $A$ ($\frac{1}{2}$, $\frac{1}{2}$ –$\frac{1}{2}$) AND $B$ (10, 0, 11)

**G**  $A$ ($\sqrt{2}$, 5, 0) AND $B$ (0, $\frac{1}{2}$, $\sqrt{3}$)   **H**  $A$ (0, 0, –2) AND $B$ (0, 0, 5)

**4**  FIND THE MIDPOINT OF THE LINE SEGMENT WHOSE END POINTS ARE:

**A**  A (0, 0, 0) AND B (4, 4, 4)   **B**  C (–2, –2, –2) AND D (2, 2, 2)

**C**  P (6, 0, 0) AND Q (0, 4, 0)   **D**  R($2\sqrt{2}$, –4, 0) AND S (–$2\sqrt{2}$, 0, –5)

**5**  SHOW THAT A (0, 4, 2), B (2, 6, 5) AND C (1, 4, 3) ARE VERTICES OF AN ISOSCELES TRIANGLE.

**6**  DETERMINE THE NATURE OF $\triangle$ ABC, USING DISTANCES, IF THE VERTICES ARE AT:

**A**  $A$ (2, –1, 7), $B$ (3, 1, 4) AND $C$ (5, 4, 5)

**B**  $A$ (0, 0, 3), $B$ (2, 8, 1) AND $C$ (–9, 6, 18)

**C**  $A$ (1, 0, –3), $B$(2, 2, 0) AND $C$ (4, 6, 6)

**D**  $A$ (5, 6, –2), $B$ (6, 12, 9) AND $C$ (2, 4, 2)

**7**  MAKE A THREE DIMENSIONAL SKETCH SHOWING EACH OF THE FOLLOWING VECTORS WITH INITIAL POINT AT THE ORIGIN.

**A**  $\vec{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$,   **B**  $\vec{b} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$,

**C**  $\vec{c} = -3\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$,   **D**  $\vec{d} = 4\mathbf{j} - 7\mathbf{k}$.

**8**  USING THE VECTORS IN QUESTION 7 ABOVE, CALCULATE EACH OF THE FOLLOWING.

**A**  $\vec{a} + \vec{b}$   **B**  $2\vec{a} - \vec{c}$   **C**  $\vec{b} + \vec{c} + \vec{d}$   **D**  $2\vec{a} - 3\vec{b} + \vec{c}$

**9**  CALCULATE THE MAGNITUDE OF EACH OF THE VECTORS IN QUESTION 8 ABOVE.

**10**  A SPHERE HAS CENTRE AT C(–1, 2, 4) AND DIAMETER AB, WHERE A IS AT (–2, 1, 3). FIND THE COORDINATES OF B, THE RADIUS AND THE EQUATION OF THE SPHERE.

**11**  DECIDE WHETHER OR NOT EACH OF THE FOLLOWING IS AN EQUATION OF A SPHERE. IF IT IS AN EQUATION OF A SPHERE, DETERMINE ITS CENTRE AND RADIUS.

**A**  $x^2 + y^2 + z^2 - 2y = 4$   **B**  $x^2 + y^2 + z^2 - x + 2y - 3z + 4 = 0$

**C**  $x^2 + y^2 + z^2 - 2x + 4y - 6z + 13 = 0$

**12**  CALCULATE THE SCALAR (DOT) PRODUCT OF EACH PAIR OF THE FOLLOWING VECTORS.

**A**  $\vec{a} = (3, 2, –4)$ AND $\vec{b} = (3, –2, 7)$   **B**  $\vec{c} = (–1, 6, 5)$ AND $\vec{d} = (10, 3, 1)$

**C**  $\vec{p} = (2, 5, 6)$ AND $\vec{q} = (6, 6, –7)$   **D**  $\vec{a} = (7, 8, 9)$ AND $\vec{b} = (5, –9, 5)$

**13**  FOR EACH PAIR OF VECTORS IN QUESTION 12 ABOVE, FIND THE COS $\theta$, WHERE $\theta$ IS THE ANGLE BETWEEN THE VECTORS.